

Last Name:
Name:
Instructor:

Math 150
Group Final (Fall 2006)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- 1 _____/4
- 2 _____/4
- 3 _____/4
- 4 _____/4
- 5 _____/4
- 6 _____/4
- 7 _____/4
- 8 _____/8
- 9 _____/8
- 10 _____/8
- 11 _____/8
- 12 _____/12
- 13 _____/4
- 14 _____/4
- 15 _____/4
- 16 _____/4
- 17 _____/4
- 18 _____/8

Total: /100

1 (4 pts.) Determine

$$\lim_{x \rightarrow 2^-} \frac{x - 4}{x^2 - 5x + 6}$$

2 (4 pts.) Determine

$$\lim_{x \rightarrow +\infty} \frac{x^2}{e^{x/4}}$$

by using L'Hôpital's Rule.

In problems 3-7 determine $f'(x)$:

3 (4 pts.)

$$f(x) = \frac{x^4 + 1}{x^4 - 1}$$

4 (4 pts.)

$$f(x) = \left(\sin \left(\frac{x}{4} \right) \right)^{1/3}$$

5 (4 pts.)

$$f(x) = x^3 \cos(x)$$

6 (4 pts.)

$$f(x) = \ln\left(\frac{x-1}{x+1}\right)$$

7 (4 pts.)

$$f(x) = \arcsin(3x)$$

8 Let

$$f(x) = \frac{1}{x^{2/3}}.$$

a) (4 pts.) Determine the linear approximation to f based at 1 (i.e., the linearization of f at 1).

b) (4 pts.) Make use of the result of part a) to approximate

$$\frac{1}{(0.7)^{2/3}}.$$

9 (8 pts.) Assume that the area of a disk is increasing at the rate of $100 \text{ cm}^2/\text{sec}$. Determine the rate at which the radius of the disk is increasing at the instant the radius is 10 cm.

10 (8 pts.) Assume that $y(x)$ is defined implicitly by the equation

$$y^3 - 4xy = 8,$$

and $y(2) = -2$. Determine $y'(2)$.

11 (8 pts.) Let

$$f(x) = \frac{1}{12}x^4 + \frac{1}{3}x^3 - 4x^2$$

Determine the intervals on which the graph of f is concave up/concave down, and the x -coordinates of the points of inflection of the graph of f .

12 Let

$$f(x) = x - 1 + \frac{1}{x - 2}$$

a) (4 pts.) Determine the vertical asymptotes for the graph of f and the relevant infinite limits. Determine $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.

b) (4 pts.) Make use of the first derivative test to determine the intervals on which f is increasing/decreasing and the points at which f has a local maximum or local minimum.

c) (2 pts.) Sketch the graph of f . Indicate the vertical asymptotes and the local extrema clearly.

d) (2 pts.) Determine the absolute maximum and the absolute minimum of f on the interval $(2, +\infty)$, provided that they exist. Justify your response if you claim that such a value does not exist.

13 (4 pts.) Compute

$$\int_1^2 \frac{d}{dx} (\sqrt{x^4 + 1}) dx$$

14 (4 pts.) Determine

$$\frac{d}{dx} \int_{\pi^2}^x \cos(\sqrt{t}) dt$$

15 (4 pts.) Determine

$$\int x^2 \sqrt{x^3 + 1} dx$$

16 (4 pts.) Compute

$$\int_{4-e^2}^{4-e} \frac{1}{x-4} dx$$

17 (4 pts.) Compute

$$\int_{-1}^0 \frac{1}{x^2+1} dx$$

18 (8 pts.) Assume that the velocity of an object moving along the x -axis is $\sin(t/3)$ at time t , and its position at $t = \pi$ is 2. Find the position of the object at any time t .

Solutions

1.

$$\lim_{x \rightarrow 2^-} \frac{x-4}{x^2-5x+6} = \lim_{x \rightarrow 2^-} \left(\frac{x-4}{x-3} \right) \left(\frac{1}{x-2} \right).$$

Since

$$\lim_{x \rightarrow 2^-} \left(\frac{x-4}{x-3} \right) = 2 > 0 \text{ and } \lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty,$$

we have

$$\lim_{x \rightarrow 2^-} \left(\frac{x-4}{x-3} \right) \left(\frac{1}{x-2} \right) = -\infty.$$

2.

$$\lim_{x \rightarrow +\infty} \frac{x^2}{e^{x/4}} = \lim_{x \rightarrow +\infty} \frac{2x}{\frac{1}{4}e^{x/4}} = \lim_{x \rightarrow \infty} \frac{8x}{e^{x/4}} = \lim_{x \rightarrow \infty} \frac{8}{\frac{1}{4}e^{x/4}} = 0.$$

3.

$$f'(x) = \frac{d}{dx} \left(\frac{x^4+1}{x^4-1} \right) = \frac{4x^3(x^4-1) - (x^4+1)(4x^3)}{(x^4-1)^2} = \frac{-8x^3}{(x^4-1)^2}.$$

4.

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\sin\left(\frac{x}{4}\right) \right)^{1/3} \\ &= \left(\frac{d}{du} u^{1/3} \Big|_{u=\sin(x/4)} \right) \left(\frac{d}{dv} \sin(v) \Big|_{v=x/4} \right) \left(\frac{d}{dx} \left(\frac{x}{4} \right) \right) \\ &= \left(\frac{1}{3 \sin^{2/3}(x/4)} \right) \left(\cos\left(\frac{x}{4}\right) \right) \left(\frac{1}{4} \right) = \frac{\cos(x/4)}{12 \sin^{2/3}(x/4)}. \end{aligned}$$

5.

$$\begin{aligned} f'(x) = \frac{d}{dx} x^3 \cos(x) &= \left(\frac{d}{dx} x^3 \right) \cos(x) + x^3 \left(\frac{d}{dx} \cos(x) \right) \\ &= 3x^2 \cos(x) - x^3 \sin(x) .. \end{aligned}$$

6.

$$f'(x) = \frac{d}{dx} \ln\left(\frac{x-1}{x+1}\right) = \frac{d}{dx} (\ln(x-1) - \ln(x+1)) = \frac{1}{x-1} - \frac{1}{x+1}.$$

7.

$$\frac{d}{dx} \arcsin(3x) = \frac{1}{\sqrt{1-9x^2}} (3) = \frac{3}{\sqrt{1-9x^2}}.$$

8.

a) We have

$$f'(x) = \frac{d}{dx}x^{-2/3} = -\frac{2}{3}x^{-5/3}.$$

Therefore,

$$f(1) = 1 \text{ and } f'(1) = -\frac{2}{3}.$$

Thus,

$$L_{-1}(x) = f(1) + f'(1)(x-1) = 1 - \frac{2}{3}(x-1).$$

b)

$$\frac{1}{0.7^{2/3}} = f(0.7) \cong L_{-1}(0.7) = 1 - \frac{2}{3}(-0.3) = 1 + 0.2 = 1.2$$

9.

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

\Rightarrow

$$100 = 2\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{100}{2\pi r}.$$

When $r = 10$,

$$\frac{dr}{dt} = \frac{100}{2\pi(10)} = \frac{5}{\pi} \text{ cm/sec.}$$

10.

$$3y^2 \frac{dy}{dx} - 4y - 4x \frac{dy}{dx} = 0$$

\Rightarrow

$$(3y^2 - 4x) \frac{dy}{dx} = 4y \Rightarrow \frac{dy}{dx} = \frac{4y}{3y^2 - 4x}.$$

Therefore,

$$y'(2) = \left. \frac{4y}{3y^2 - 4x} \right|_{x=2, y=-2} = -2.$$

11. We have

$$f'(x) = \frac{d}{dx} \left(\frac{1}{12}x^4 + \frac{1}{3}x^3 - 4x^2 \right) = \frac{1}{3}x^3 - x^2 - 8x,$$

and

$$f''(x) = \frac{d}{dx} \left(\frac{1}{3}x^3 - x^2 - 8x \right) = x^2 - 2x - 8 = (x-4)(x+2).$$

Therefore,

$$f''(x) = 0 \Leftrightarrow x = -2 \text{ or } x = 4.$$

We have

$$f''(x) > 0 \text{ if } x < -2. \quad f''(x) < 0 \text{ if } -2 < x < 4 \text{ and } f''(x) > 0 \text{ if } x > 4.$$

Therefore, the graph of f is concave up on $(-\infty, -2]$, concave down on $[-2, 4]$, and concave up on $[4, +\infty)$. The x -coordinates of the inflection points of the graph of f are -2 and 4 .

12.

a) The vertical asymptotes is the line $x = 2$. We have

$$\lim_{x \rightarrow 2^-} f(x) = -\infty, \quad \lim_{x \rightarrow 2^+} f(x) = +\infty..$$

We have

$$\lim_{x \rightarrow +\infty} f(x) = +\infty \text{ and } \lim_{x \rightarrow -\infty} f(x) = -\infty.$$

b) We have

$$f'(x) = 1 - \frac{1}{(x-2)^2} = \frac{x^2 - 4x + 3}{(x-2)^2} = \frac{(x-1)(x-3)}{(x-2)^2}$$

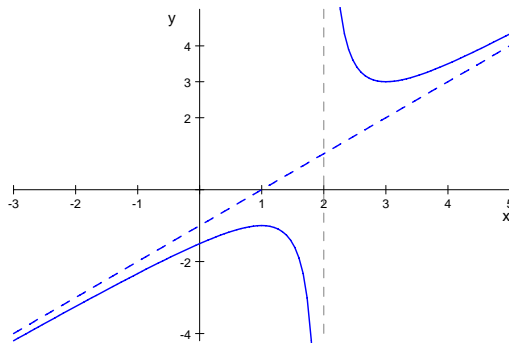
Therefore,

$$f'(x) = 0 \iff x = 1 \text{ or } x = 3.$$

x		1		2		3	
$f'(x)$	+	0	-	undef	-	0	+
f	incr.	loc. max.	decr..	undef.	decr.	loc. min	incr.

The function is increasing on $(-\infty, 1]$, decreasing on $[1, 2)$, decreasing on $(2, 3]$, and increasing on $(3, +\infty)$. Thus, f has a local maximum at 1 and a local minimum at 3.

c)



d) The absolute minimum of f on $(2, +\infty)$ is $f(3) = 3$. The function does not have an absolute maximum on $(-2, 2)$ since

$$\lim_{x \rightarrow 2^+} f(x) = +\infty \text{ (and } \lim_{x \rightarrow +\infty} f(x) = +\infty).$$

13.

$$\int_1^2 \frac{d}{dx} (\sqrt{x^4 + 1}) dx = \sqrt{x^4 + 1} \Big|_1^2 = \sqrt{16} - \sqrt{2}$$

14.

$$\frac{d}{dx} \int_{\pi^2}^x \cos(\sqrt{t}) dt = \cos(\sqrt{x}).$$

15. We set $u = x^3 + 1$ so that $du = 3x^2 dx$. Thus,

$$\int x^2 \sqrt{x^3 + 1} dx = \frac{1}{3} \int \sqrt{u} du = \frac{1}{3} \left(\frac{u^{3/2}}{3/2} \right) = \frac{2}{9} (x^3 + 1)^{3/2}$$

16.

$$\int_{4-e^2}^{4-e} \frac{1}{x-4} dx = \ln(|x-4|) \Big|_{4-e^2}^{4-e} = \ln(|-e|) - \ln(|-e^2|) = \ln(e) - \ln(e^2) = 1 - 2 = -1$$

17,

$$\int_{-1}^0 \frac{1}{x^2+1} dx = \arctan(x) \Big|_{-1}^0 = \arctan(0) - \arctan(-1) = \frac{\pi}{4}$$

18. We have

$$f(t) = \int v(t) dt = \int \sin\left(\frac{t}{3}\right) dt = -3 \cos\left(\frac{t}{3}\right) + C.$$

Therefore,

$$f(\pi) = 2 \Leftrightarrow -3 \cos\left(\frac{\pi}{3}\right) + C = 2 \Leftrightarrow -\frac{3}{2} + C = 2 \Leftrightarrow C = \frac{7}{2}.$$

Thus,

$$f(t) = -3 \cos\left(\frac{t}{3}\right) + \frac{7}{2}.$$