Last Name:
Name:
Instructor:

Math 150
Group Final (Spring 2007)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- **1**________/4
- **2**________/4
- **3**________/4
- **4**________/4
- **5**________/4
- **6**________/4
- **7**________/4
- **8**________/8
- **9**________/8
- **10**________/8
- **11**________/8
- **12**________/12
- **13**________/4
- **14**________/4
- **15**________/4
- **16**________/4
- **17**________/4
- **18**________/8

Total: **/100**

1 (4 pts.) Determine

$$\lim_{x \to 4-} \frac{\left|x^2 - 16\right|}{x^2 + x - 20}$$

2 (4 pts.) Determine

$$\lim_{x \to 0+} x^{1/4} \ln(x) .$$

by using L'Hospital's Rule.

3 (4 pts.) Determine

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+9}}{4x-3}$$

In problems 4-7 determine $f'(x)$. Carry out the obvious simplifications.

4 (4 pts.)

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

5 (4 pts.)

$$f\left(x\right)=e^{-(x-1)^2/2}$$

6 (4 pts.)

$$f\left(x\right)=\arctan\left(x^3\right)$$

7 (4 pts.)

$$f\left(x\right)=\frac{x^3-8}{x^2+4}$$

8. Let

$$f(x) = \sqrt{25 - x^2}.$$

a) (4 pts.) Determine the linearization of f at 3 (i.e., the linear approximation to f based at 3).

b) (4 pts.) Make use of the result of part a) to approximate $f(2.8)$

9 (8 pts.) Assume that a ladder which is 9 feet long is leaning against a wall and its base is sliding away from the wall at the rate of 1 ft/sec. Determine the rate at which the angle between the ladder and the ground is changing at the instant the base of the ladder is 3 feet from the wall.

10 (8 pts.) Assume that $y\,(x)$ is defined implicitly by the equation

$$y^3 - 9y - 3x = 0$$

and $y\,(0) = 3$. Determine $y'\,(0)$.

11 (8 pts.) Let

$$f(x) = \frac{1}{12}x^4 - \frac{1}{3}x^3 - \frac{3}{2}x^2$$

Determine the intervals on which the graph of f is concave up/concave down, and the x-coordinates of the points of inflection of the graph of f.

12 Let

$$f(x) = \frac{e^x}{x}.$$

a) (4 pts.) Determine the vertical asymptotes for the graph of f and the relevant infinite limits. Determine $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

b) (4 pts.) Make use of the first derivative test to determine the intervals on which f is increasing/decreasing and the points at which f has a local maximum or local minimum.

c) (2 pts.) Sketch the graph of f. Indicate the vertical asymptotes and the local extrema clearly.

d) (2 pts.) Determine the absolute maximum and the absolute minimum of f on the interval $(0, +\infty)$, provided that they exist. Justify your response if you claim that such a value does not exist.

13 (4 pts.) Compute

$$\int_{\pi/6}^{\pi/3} \frac{d}{dx}\left(\sin^2(x)\right) dx$$

14 (4 pts.) Determine

$$\frac{d}{dx}\int_{x}^{1} \frac{1}{\sqrt{9-t^2}\sqrt{4-t^2}} dt$$

15 (4 pts.) Determine

$$\int \frac{x}{\sqrt{9-x^2}}\, dx$$

16 (4 pts.) Compute

$$\int_{\sqrt{\ln(4)}}^{\sqrt{\ln(10)}} xe^{-x^2}\,dx$$

17 (4 pts.) Compute

$$\int \frac{x}{x^2+9}dx$$

18 (8 pts.) Assume that the velocity of an object moving along a line is

$$\frac{t}{\left(t^2+1\right)^2}$$

at time t, and its position at $t = 1$ is 2. Find the position of the object at any time t.

Solutions

1.

$$\begin{aligned}\lim_{x\to 4-}\frac{\left|x^2-16\right|}{x^2+x-20}=\lim_{x\to 4-}\frac{-\left(x^2-16\right)}{x^2+x-20} &= -\lim_{x\to 4-}\frac{(x-4)(x+4)}{(x+5)(x-4)}\\ &= -\lim_{x\to 4-}\frac{x+4}{(x+5)}=-\frac{8}{9}.\end{aligned}$$

2.

$$\begin{aligned}\lim_{x\to 0+}x^{1/4}\ln(x)=\lim_{x\to 0+}\frac{\ln(x)}{x^{-1/4}} &= \lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{4}x^{-5/4}}\\ &= -4\lim_{x\to 0+}x^{1/4}=0.\end{aligned}$$

3.

$$\lim_{x\to -\infty}\frac{\sqrt{x^2+9}}{4x-3}=\lim_{x\to -\infty}\frac{|x|\sqrt{1+\frac{9}{x^2}}}{x\left(4-\frac{3}{x}\right)}=\lim_{x\to -\infty}\frac{-x\sqrt{1+\frac{9}{x^2}}}{x\left(4-\frac{3}{x}\right)}=-\lim_{x\to -\infty}\frac{\sqrt{1+\frac{9}{x^2}}}{4-\frac{3}{x}}=-\frac{1}{4}.$$

4.

$$\begin{aligned}f'(x)=\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+1}}\right) &= \frac{\sqrt{x^2+1}-x\left(\frac{x}{\sqrt{x^2+1}}\right)}{x^2+1}.\\ &= \frac{x^2+1-x^2}{\left(x^2+1\right)^{3/2}}=\frac{1}{\left(x^2+1\right)^{3/2}}\end{aligned}$$

5.

$$\begin{aligned}f'(x)=\frac{d}{dx}\left(e^{-(x-1)^2/2}\right) &= e^{-(x-1)^2/2}\left(-\frac{1}{2}\frac{d}{dx}(x-1)^2\right)\\ &= e^{-(x-1)^2/2}\left(-\frac{1}{2}\left(2(x-1)\right)\right)=(1-x)\,e^{-(x-1)^2/2}\end{aligned}$$

6.

$$\begin{aligned}f'(x)=\frac{d}{dx}\arctan\left(x^3\right) &= \left(\frac{d}{du}\arctan(u)\Big|_{u=x^3}\right)\left(\frac{d}{dx}\left(x^3\right)\right)\\ &= \left(\frac{1}{1+u^2}\Big|_{u=x^3}\right)\left(3x^2\right)=\frac{3x^2}{1+x^6}.\end{aligned}$$

7.

$$\begin{aligned}f'(x)=\frac{d}{dx}\left(\frac{x^3-8}{x^2+4}\right) &= \frac{3x^2\left(x^2+4\right)-\left(x^3-8\right)(2x)}{\left(x^2+4\right)^2}\\ &= \frac{x^4+12x^2+16x}{\left(x^2+4\right)^2}\end{aligned}$$

8.
a) We have

$$f'(x) = \frac{d}{dx}\sqrt{25-x^2} = \left(\frac{1}{2\sqrt{25-x^2}}\right)(-2x) = -\frac{x}{\sqrt{25-x^2}}.$$

Therefore,

$$f(3) = \sqrt{25-9} = \sqrt{16} = 4 \text{ and } f'(3) = -\frac{3}{4}..$$

Thus,

$$L_3(x) = f(3) + f'(3)(x-3) = 4 - \frac{3}{4}(x-3).$$

b)

$$f(2.8) \cong L_3(2.8) = 4 - \frac{3}{4}(-0.2) = 4 + \frac{0.3}{2} = 4.15.$$

9. If we label the distance of the bottom of the ladder from the wall by x and the angle between the ladder and the ground by θ, we have

$$\cos(\theta) = \frac{x}{9}.$$

Therefore,

$$-\sin(\theta)\frac{d\theta}{dt} = \frac{1}{9}\frac{dx}{dt} = \left(\frac{1}{9}\right)(3) = \frac{1}{3}.$$

Thus,

$$\frac{d\theta}{dt} = -\frac{1}{3\sin(\theta)}.$$

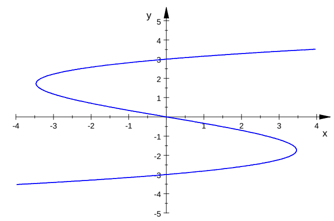
At the instant $x = 3$ we have

$$\sin(\theta) = \frac{\sqrt{81-9}}{9} = \frac{\sqrt{72}}{9}.$$

Therefore, at that instant

$$\frac{d\theta}{dt} = -\frac{1}{3\left(\frac{\sqrt{72}}{9}\right)} = -\frac{3}{\sqrt{72}}.$$

10.



$$3y^2\frac{dy}{dx}-9\frac{dy}{dx}-3=0$$

$\Rightarrow$

$$\left(3y^2-9\right)\frac{dy}{dx}=3\Rightarrow\frac{dy}{dx}=\frac{3}{3y^2-9}=\frac{1}{y^2-3}.$$

If $y\left(0\right)=3$,

$$y'\left(0\right)=\left.\frac{1}{y^2-3}\right|_{y=3}=\frac{1}{9-3}=\frac{1}{6}.$$

11. We have

$$f'\left(x\right)=\frac{d}{dx}\left(\frac{1}{12}x^4-\frac{1}{3}x^3-\frac{3}{2}x^2\right)=\frac{1}{3}x^3-x^2-3x$$

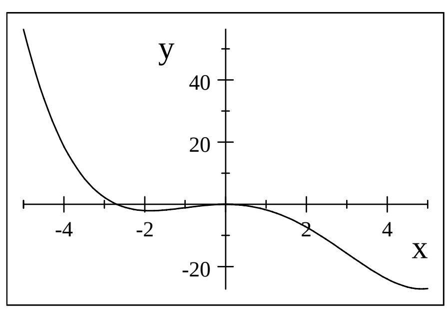
and

$$f''\left(x\right)=\frac{d}{dx}\left(\frac{1}{3}x^3-x^2-3x\right)=x^2-2x-3=\left(x+1\right)\left(x-3\right).$$

Therefore,

$$f''\left(x\right)>0 \text{ if } x<-1,\ f'\left(x\right)<0 \text{ if } -1<x<3 \text{ and } f''\left(x\right)>0 \text{ if } x>3.$$

Thus, the graph of f is concave up on $(-\infty,-1]$, concave down on $[-1,3]$ and concave up on $[3,+\infty)$. The x-coordinates of the points of inflection of the graph of f are -1 and 3.



12.
a) The vertical asymptotes is the y-axis $x=0$. We have

$$\lim_{x\to 0-}\frac{e^x}{x}=-\infty,\ \lim_{x\to 0+}\frac{e^x}{x}=+\infty,\ \lim_{x\to -\infty}\frac{e^x}{x}=0 \text{ and } \lim_{x\to +\infty}\frac{e^x}{x}=+\infty.$$

b) We have

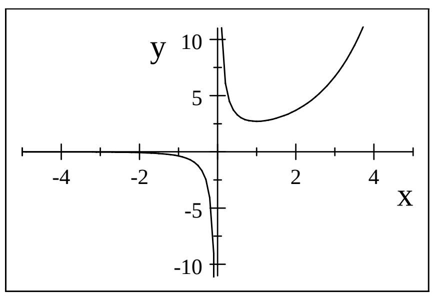
$$f'(x)=\frac{e^x x-e^x}{x^2}=\frac{e^x(x-1)}{x^2}.$$

Therefore,

$$f'(x)=0\Longleftrightarrow x=.1$$

x		0		1	
$f'(x)$	$-$	undef.	$-$	0	$+$
f	decr.	undef.	decr.	loc. min	incr.

The function is decreasing on $(-\infty,0)$, decreasing on $(0,1]$, and increasing on $[1,+\infty)$. Thus, f has a local minimu at 1.
c)



d) The absolute minimum of f on $(0,+\infty)$ is $f(1)=e$.The function does not have an absolute maximum on $(-,+\infty)$ since

$$\lim_{x\to 0+}f(x)=+\infty \text{ (and } \lim_{x\to +\infty}f(x)=+\infty).$$

13.

$$\int_{\pi/6}^{\pi/3}\frac{d}{dx}\left(\sin^2(x)\right)dx=\sin^2\left(\frac{\pi}{3}\right)-\sin^2\left(\frac{\pi}{6}\right)=\left(\frac{\sqrt{3}}{2}\right)^2-\left(\frac{1}{2}\right)^2=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}.$$

14.

$$\frac{d}{dx}\int_x^1\frac{1}{\sqrt{9-t^2}\sqrt{4-t^2}}dt=-\frac{1}{\sqrt{9-x^2}\sqrt{4-x^2}}.$$

15. We set $u=9-x^2$ so that $du=-2xdx$. Thus,

$$\int\frac{x}{\sqrt{9-x^2}}dx=-\frac{1}{2}\int\frac{1}{\sqrt{u}}du=-\frac{1}{2}\int u^{-1/2}du=-\frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right)=-\sqrt{9-x^2}.$$

16. We set $u = -x^2$ so that $du = -2xdx$. Therefore,

$$\int_{\sqrt{\ln(4)}}^{\sqrt{\ln(10)}} xe^{-x^2}\,dx = -\frac{1}{2}\int_{-\ln(4)}^{-\ln(10)} e^u du = \quad -\frac{1}{2}\left(e^{-\ln(10)} - e^{-\ln(4)}\right)$$
$$= \quad -\frac{1}{2}\left(\frac{1}{10} - \frac{1}{4}\right) = \frac{3}{40}$$

17. We set $u = x^2 + 9$ so that $du = 2xdx$. Therefore,

$$\int \frac{x}{x^2 - 9}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(|u|) = \frac{1}{2}\ln\left(\left|x^2 - 9\right|\right)$$

18. We have

$$f(t) = \int \frac{t}{(t^2+1)^2}dt = -\frac{1}{2(t^2+1)} + C.$$

Therefore,

$$f(1) = 2 \Leftrightarrow -\frac{1}{2(1+1)} + C = 2 \Leftrightarrow -\frac{1}{4} + C = 2 \Leftrightarrow C = \frac{9}{4}$$

Thus,

$$f(t) = -\frac{1}{2(t^2+1)} + \frac{9}{4}..$$