

Last Name:  
First Name:  
Instructor:

Math 151  
Group Final (Fall 2006)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- 1            /5
- 2            /5
- 3            /5
- 4            /5
- 5            /5
- 6            /10
- 7            /10
- 8            /10
- 9            /10
- 10           /5
- 11           /10
- 12           /10
- 13           /5
- 14           /5

Total:            /100

**1** (5 pts.) Determine

$$\int x \cos\left(\frac{x}{2}\right) dx$$

**2** (5 pts.) Determine

$$\int \frac{x-1}{x^2-7x+12} dx$$

**3** (5 pts.) Determine

$$\int \sin^2(3x) dx.$$

**4** (5 pts.) Determine whether the improper integral

$$\int_0^{\infty} \frac{x}{(x^2 + 1)^2} dx$$

converge or diverges, and its value in case it converges.

5 (5 pts.) Use a comparison test to determine whether the improper integral

$$\int_1^{\infty} \frac{\sin^2(x)}{e^x} dx$$

converges or diverges.

**6** (10 pts.) Determine the volume of the solid that is obtained by revolving the region between the graph of

$$f(x) = \sqrt{\frac{x}{x^2 + 4}}$$

and the interval  $[\sqrt{e^2 - 4}, \sqrt{e^3 - 4}]$  about the  $x$ -axis.

**7** (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem

$$\frac{dy}{dt} + y(t) = e^{-t}, \quad y(0) = 2.$$

**8** (10 pts.) Determine the solution of the initial value problem

$$\frac{dy}{dt} = \sin(t) y^2, \quad y(\pi) = 4$$

9. Let

$$r = f(\theta) = 2 - 2\sin(\theta).$$

a) (5 pts.) Sketch the graph of  $r = f(\theta)$  in the Cartesian  $\theta r$ -plane on the interval  $[0, 2\pi]$ . Indicate the values of  $\theta$  at which  $f(\theta) = 0$  and the points at which  $f$  attains a local maximum or minimum value.

b) (5 pts.) Sketch the graph of  $r = f(\theta)$ , where  $0 \leq \theta \leq 2\pi$ , as a polar equation in the  $xy$ -plane (i.e.,  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ ).

**10** (5 pts.) Determine whether the infinite series

$$\sum_{n=0}^{\infty} \frac{n}{2^n}$$

converges or diverges.

**11** (10 pts.) Determine whether the infinite series

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{(n^2 + 1)^{2/3}}$$

converges absolutely, converges conditionally or diverges.

**12** (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{n^3}{3^n} (x-4)^n.$$

(You need not investigate the series at the endpoints of the interval.)

**13** (5 pts.) Let  $f(x) = \ln(x)$ . Determine the part of the Taylor series for  $f$  based at 2 up to the term that has  $(x-2)^3$ .

14 (5 pts.) Let

$$F(x) = \int_0^x \frac{1}{1+t^4} dt.$$

Determine the Maclaurin series for  $F$  up to the term that has  $x^{13}$

Hint: The starting point can be the geometric series.

### Solutions

1. We set  $u = x$  and  $dv = \cos(x/2) dx$  so that

$$du = dx \text{ and } v = \int \cos\left(\frac{x}{2}\right) dx = 2 \sin\left(\frac{x}{2}\right).$$

Thus,

$$\begin{aligned} \int x \cos\left(\frac{x}{2}\right) dx &= \int u dv = uv - \int v du \\ &= x \left(2 \sin\left(\frac{x}{2}\right)\right) - 2 \int \sin\left(\frac{x}{2}\right) dx \\ &= 2x \sin\left(\frac{x}{2}\right) + 4 \cos\left(\frac{x}{2}\right) \end{aligned}$$

2.

$$\frac{x-1}{x^2-7x+12} = \frac{x-1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

$\Leftrightarrow$

$$x-1 = A(x-4) + B(x-3).$$

Set  $x = 3$ :

$$2 = -A \Rightarrow A = -2.$$

Set  $x = 4$ :

$$3 = B.$$

Thus,

$$\begin{aligned} \int \frac{x-1}{x^2-7x+12} dx &= -2 \int \frac{1}{x-3} dx + 3 \int \frac{1}{x-4} dx \\ &= -2 \ln(|x-3|) + 3 \ln(|x-4|). \end{aligned}$$

3.

$$\int \sin^2(3x) dx = \int \frac{1 - \cos(6x)}{2} dx = \frac{1}{2}x - \frac{1}{12} \sin(6x).$$

4. We have

$$\int \frac{x}{(x^2+1)^2} dx = -\frac{1}{2(x^2+1)},$$

so that

$$\int_0^b \frac{x}{(x^2+1)^2} dx = -\frac{1}{2(b^2+1)} + \frac{1}{2}.$$

Therefore,

$$\int_0^\infty \frac{x}{(x^2+1)^2} dx = \frac{1}{2}$$

5. We have

$$0 \leq \frac{\sin^2(x)}{e^x} \leq \frac{1}{e^x}$$

and

$$\int_1^\infty e^{-x} dx = e^{-1}.$$

Therefore,

$$\int_1^\infty \frac{\sin^2(x)}{e^x} dx$$

converges.

6.

$$\pi \int_{\sqrt{e^2-4}}^{\sqrt{e^3-4}} \frac{x}{x^2+4} dx.$$

We have

$$\pi \int \frac{x}{x^2+4} dx = \frac{\pi}{2} \ln(x^2+4).$$

Thus,

$$\pi \int_{\sqrt{e^2-4}}^{\sqrt{e^3-4}} \frac{x}{x^2+4} dx = \frac{\pi}{2} \ln(e^3) - \frac{\pi}{2} \ln(e^2) = \frac{\pi}{2}.$$

7. The integrating factor is  $e^t$ :

$$e^t \left( \frac{dy}{dt} + y(t) \right) = e^t e^{-t}$$

$\Leftrightarrow$

$$\frac{d}{dt} (e^t y) = 1$$

$\Rightarrow$

$$e^t y(t) = t + C \Rightarrow y(t) = te^{-t} + Ce^{-t}.$$

We have

$$y(0) = 2 \Leftrightarrow C = 2.$$

Therefore,

$$y(t) = te^{-t} + 2e^{-t}.$$

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$$\begin{aligned}\frac{dy}{dt} = \sin(t) y^2 &\Rightarrow \frac{1}{y^2} \frac{dy}{dt} = \sin(t) \\ &\Rightarrow \int \frac{1}{y^2} \frac{dy}{dt} dt = \int \sin(t) dt \\ &\Rightarrow \int \frac{1}{y^2} dy = -\cos(t) + C \\ &\Rightarrow -\frac{1}{y} = -\cos(t) + C \\ &\Rightarrow y(t) = \frac{1}{\cos(t) - C}.\end{aligned}$$

We have

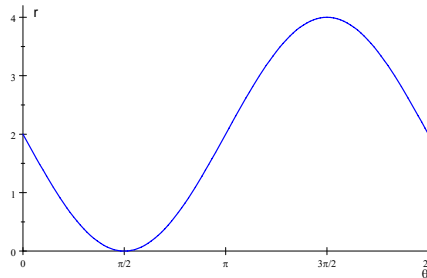
$$y(\pi) = 4 \Leftrightarrow \frac{1}{-1 - C} = 4 \Leftrightarrow -\frac{1}{4} = C + 1 \Leftrightarrow C = -\frac{5}{4}.$$

Therefore,

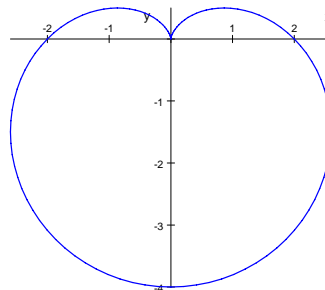
$$y(t) = \frac{1}{\cos(t) + \frac{5}{4}}.$$

9.

a)



b)



10.

$$\lim_{n \rightarrow \infty} \left(\frac{n}{2^n}\right)^{1/n} = \lim_{n \rightarrow \infty} \frac{n^{1/n}}{2} = 12 < 1.$$

Therefore the series converges.

11. Since

$$\frac{1}{(n^2 + 1)^{2/3}} < \frac{1}{n^{4/3}},$$

and

$$\sum \frac{1}{n^{4/3}}$$

converges, the series converges absolutely.

12.

$$\lim_{n \rightarrow \infty} \left(\frac{n^3}{3^n} |x - 4|^n\right)^{1/n} = |x - 4| \lim_{n \rightarrow \infty} \frac{n^{3/n}}{3} = \frac{1}{3} |x - 4| < 1$$

$\Leftrightarrow$

$$|x - 4| < 3 \Leftrightarrow 1 < x < 7.$$

Thus, the radius of convergence of the series is 3 and the open interval of convergence of the series is (1, 7).

13.

$$\begin{aligned} f(x) &= \ln(x) \Rightarrow f(2) = \ln(2), \\ f'(x) &= \frac{1}{x} \Rightarrow f'(2) = \frac{1}{2}, \\ f''(x) &= -x^{-2} \Rightarrow f''(2) = -\frac{1}{4}, \\ f'''(x) &= 2x^{-3} \Rightarrow f'''(2) = \frac{1}{4}. \end{aligned}$$

Therefore,

$$\ln(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 + \dots$$

14.

$$\frac{1}{1+t^4} = 1 - t^4 + t^8 - t^{12} + \dots$$

Therefore,

$$\begin{aligned} F(x) &= \int_0^x \frac{1}{1+t^4} dt = \int_0^x (1 - t^4 + t^8 - t^{12} + \dots) dt \\ &= x - \frac{1}{5}x^5 + \frac{1}{9}x^9 - \frac{1}{13}x^{13} + \dots \end{aligned}$$