Last Name:
First Name:
Instructor:

Math 151
Group Final (Spring 2006)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- **1** /5
- **2** /10
- **3** /5
- **4** /5
- **5** /5
- **6** /10
- **7** /10
- **8** /10
- **9** /10
- **10** /5
- **11** /10
- **12** /5
- **13** /5
- **14** /5

Total: **/100**

1 (5 pts.) Determine

$$\int \sin^3(x)\cos^2(x)\,dx$$

2 (10 pts.) Determine

$$\int \frac{3x+14}{x^2+x-6}dx$$

3 (5 pts.) Determine

$$\int x \cosh\left(\frac{x}{4}\right) dx$$

4 (5 pts.) Determine whether the improper integral

$$\int_{\sqrt{\ln(2)}}^{\infty} xe^{-x^2}\,dx$$

converge or diverges, and its value in case it converges (express your response in a simplified form).

5 (5 pts.) Use the comparison test to determine whether the improper integral

$$\int_0^1 \frac{e^x}{x^{3/2}}\,dx$$

converges or diverges (if you claim that the integral converges, you do not have to evaluate the value of the integral).

6 (10 pts.) Determine the volume of the solid that is obtained by revolving the region between the graph of

$$f\left(x\right)=\frac{1}{x^{2}+4}$$

and the interval $[0,2]$ about the vertical axis.

7 (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem

$$\frac{dy}{dt} + \frac{y\left(t\right)}{t} = \sin\left(t\right), \; y\left(\pi\right) = 3$$

Assume that $t > 0$.

8 (10 pts.) Determine the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y^2}{1+x^2},\ y\left(1\right) = -\frac{2}{\pi}$$

9. Let

$$r = f(\theta) = \cos(2\theta).$$

a) (5 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a local maximum or minimum value.

b) (5 pts.) Sketch the graph of $r = f(\theta)$, where $0 \leq \theta \leq 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

10 (5 pts.) Determine whether the infinite series

$$\sum_{n=0}^{\infty} \frac{n!}{10^n}$$

converges or diverges.

11 (10 pts.) Determine whether the infinite series

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n \ln^2(n)}$$

converges absolutely, converges conditionally or diverges.

12 (5 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n^2} (x-4)^n .$$

(You need not investigate the series at the endpoints of the interval.)

13 (5 pts.) Let

$$F(x) = \int_0^x t^2 e^{-t^2}\, dt.$$

Determine the first 4 nonzero terms of the Maclaurin series for F and the coefficient of x^{2n+3}, where n is an arbitrary nonnegative integer.
Hint: Start with the Maclaurin series for the natural exponential function.

14 (5 pts.)
Let

$$f(x) = \begin{cases} \dfrac{\sin(x) - x}{x^3} & \text{if} \quad x \neq 0, \\ \\ -\dfrac{1}{6} & \text{if} \quad x = 0. \end{cases}$$

Determine the first 3 nonzero terms of the Maclaurin series for f.and the coefficient of x^{2n-2}, where n is an arbitrary positive integer.
Hint: Start with the Maclaurin series for sine.

Solutions

1.

$$\int \sin^3(x)\cos^2(x)\,dx = \int \sin^2(x)\cos^2(x)\sin(x)\,dx = \int \left(1-\cos^2(x)\right)\cos^2(x)\sin(x)\,dx.$$

We set $u = \cos(x)$ so that $du = -\sin(x)\,dx$:

$$\begin{aligned} \int \left(1-\cos^2(x)\right)\cos^2(x)\sin(x)\,d &= -\int \left(1-u^2\right)u^2du \\ &= -\int \left(u^2-u^4\right)du \\ &= -\frac{1}{3}u^3+\frac{1}{5}u^5 \\ &= -\frac{1}{3}\cos^3(x)+\frac{1}{5}\cos^5(x). \end{aligned}$$

2. We have

$$\frac{3x+14}{x^2+x-6} = \frac{3x+14}{(x+3)(x-2)} = \frac{A}{x+3}+\frac{B}{x-2}$$

$\Leftrightarrow$

$$3x+14 = A(x-2)+B(x+3)$$

We set $x=-3$:

$$5=-5A \Rightarrow A=-1.$$

We set $x=2$:

$$20=5B \Rightarrow B=4.$$

Thus,

$$\int \frac{3x+14}{x^2+x-6}dx = \int \left(-\frac{1}{x+3}+\frac{4}{x-2}\right)dx = -\ln(|x+3|)+4\ln(|x-2|).$$

3. We set $u = x$ and $dv = \cosh(x/4)$ so that

$$du = dx \text{ and } v = \int \cosh\left(\frac{x}{4}\right) dx = 4\sinh\left(\frac{x}{4}\right).$$

Thus,

$$\begin{aligned}\int x\cosh\left(\frac{x}{4}\right) dx &= \int u dv \\ &= uv - \int v du \\ &= x\left(4\sinh\left(\frac{x}{4}\right)\right) - \int 4\sinh\left(\frac{x}{4}\right) dx \\ &= 4x\sinh\left(\frac{x}{4}\right) - 16\cosh\left(\frac{x}{4}\right).\end{aligned}$$

4. We set $u = -x^2$ so that $du = -2xdx$. Thus,

$$\int xe^{-x^2} dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}.$$

Therefore,

$$\begin{aligned}\int_{\sqrt{\ln(2)}}^{B} xe^{-x^2} dx = -\frac{1}{2}e^{-x^2}\Big|_{\sqrt{\ln(2)}}^{B} &= -\frac{1}{2}e^{-B^2} + \frac{1}{2}e^{-\ln(2)} \\ &= -\frac{1}{2}e^{-B^2} + \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2}e^{-B^2} + \frac{1}{4}.\end{aligned}$$

Therefore,

$$\lim_{B\to\infty}\int_{\sqrt{\ln(2)}}^{B} xe^{-x^2} dx = \lim_{B\to\infty}\left(-\frac{1}{2}e^{-B^2} + \frac{1}{4}\right) = \frac{1}{4}.$$

Thus, the given improper integral converges and

$$\int_{\sqrt{\ln(2)}}^{\infty} xe^{-x^2} dx = \frac{1}{4}.$$

5. If $0 < x \leq 1$ we have

$$\frac{e^x}{x^{3/2}} \geq \frac{e^0}{x^{3/2}} = \frac{1}{x^{3/2}} > 0$$

and

$$\int_0^1 \frac{1}{x^{3/2}} dx$$

diverges. By the comparoson test,

$$\int_0^1 \frac{e^x}{x^{3/2}} dx$$

diverges as well.

6. The volume is

$$\int_0^2 2\pi x f(x)\,dx = 2\pi \int_0^2 \frac{x}{x^2+4}dx \quad = \quad \pi\left(\ln\left(x^2+4\right)\Big|_0^2\right)$$
$$= \quad \pi\left(\ln(8) - \ln(4)\right) = \pi \ln(2).$$

7. The integrating factor is

$$e^{\int \frac{1}{t}dt} = e^{\ln(t)} = t.$$

Thus,

$$t\left(\frac{dy}{dt} + \frac{y(t)}{t}\right) = t\sin(t)$$

$\Rightarrow$

$$\frac{d}{dt}\left(ty(t)\right) = t\sin(t)$$

$\Rightarrow$

$$ty(t) = \int t\sin(t)\,dt = \sin(t) - t\cos(t) + C$$

$\Rightarrow$

$$y(t) = \frac{1}{t}\sin(t) - \cos(t) + \frac{C}{t}$$

We have $y(\pi) = 3$ iff

$$3 = 1 + \frac{C}{\pi} \Leftrightarrow C = 2\pi.$$

Therefore, the solution of the initial value problem is

$$y(t) = \frac{1}{t}\sin(t) - \cos(t) + \frac{2\pi}{t}.$$

8.

$$\begin{aligned}
\frac{dy}{dx} = \frac{y^2}{1+x^2} \quad &\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} = \frac{1}{1+x^2} \\
&\Rightarrow \quad \int \frac{1}{y^2}\frac{dy}{dx}dx = \int \frac{1}{1+x^2}dx \\
&\Rightarrow \quad \int y^{-2}dy = \arctan(x) + C \\
&\Rightarrow \quad -\frac{1}{y} = \arctan(x) + C \\
&\Rightarrow \quad y = -\frac{1}{\arctan(x) + C}
\end{aligned}$$

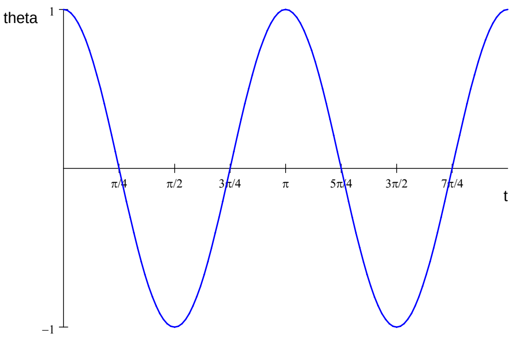
We have

$$\begin{aligned}
y(1) = -\frac{2}{\pi} \Leftrightarrow -\frac{1}{\arctan(1)+C} = -\frac{2}{\pi} \quad &\Leftrightarrow \quad .\arctan(1) + C = \frac{\pi}{2} \\
&\Leftrightarrow \quad \frac{\pi}{4} + C = \frac{\pi}{2} \Leftrightarrow C = \frac{\pi}{4}.
\end{aligned}$$

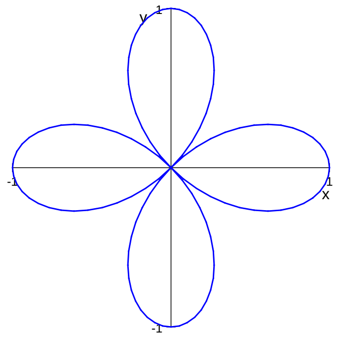
Therefore, the solution of the initial-value problem is

$$y(t) = -\frac{1}{\arctan(x) + \frac{\pi}{4}}$$

9.
a)



b)



10.

$$\lim_{n\to\infty}\frac{\dfrac{(n+1)!}{10^{n+1}}}{\dfrac{n!}{10^n}}=\lim_{n\to\infty}\left(\frac{n+1}{10}\right)=+\infty.$$

By the ratio test, the series diverges.

11. We have

$$\int_2^b \frac{1}{x\ln^2(x)}dx=\int_{\ln(2)}^{\ln(b)}\frac{1}{u^2}du=-\frac{1}{u}\Big|_{\ln(2)}^{\ln(b)}=-\frac{1}{\ln(b)}+\frac{1}{\ln(2)}.$$

Therefor,

$$\int_2^b \frac{1}{x\ln^2(x)}dx=\lim_{b\to\infty}\left(-\frac{1}{\ln(b)}+\frac{1}{\ln(2)}\right)=\frac{1}{\ln(2)}.$$

By the integral test, the series converges absolutely.

12.

$$\lim_{n\to\infty}\left|\frac{2^n}{n^2}(x-4)^n\right|^{1/n} = |x-4|\lim_{n\to\infty}\frac{2}{n^{2/n}} = 2\,|x-4|\,.$$

By the root test, the series converges absolutely if $|x-4|<1/2$ and diverges if $|x-4|>1/2$. Thus, the radius of convergence is $1/2$ and the open interval of convergence is

$$\left(4-\frac{1}{2}, 4+\frac{1}{2}\right) = \left(3\frac{1}{2}, 4\frac{1}{2}\right).$$

13. We have

$$\begin{aligned} t^2e^{-t^2} &= t^2\left(1-t^2+\frac{1}{2}\left(-t^2\right)^2+\frac{1}{3!}\left(-t^2\right)^3+\cdots+\frac{1}{n!}\left(-t^2\right)^n+\cdots\right) \\ &= t^2-t^4+\frac{1}{2}t^6-\frac{1}{3!}t^8+\cdots+(-1)^n\frac{1}{n!}t^{2n+2}+\cdots \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^x t^2e^{-t^2}dt &= \int_0^x \left(t^2-t^4+\frac{1}{2}t^6-\frac{1}{3!}t^8+\cdots+(-1)^n\frac{1}{n!}t^{2n+2}+\cdots\right)dt \\ &= \frac{x^3}{3}-\frac{x^5}{5}+\frac{1}{2}\left(\frac{x^7}{7}\right)-\frac{1}{3!}\left(\frac{x^9}{9}\right)+\cdots+(-1)^n\frac{1}{n!}\left(\frac{x^{2n+3}}{2n+3}\right)+\cdots \\ &= \frac{1}{3}x^3-\frac{1}{5}x^5+\frac{1}{14}x^7-\frac{1}{54}x^9+\cdots+(-1)^n\frac{1}{n!\,(2n+3)}x^{2n+3}+\cdots \end{aligned}$$

14. We have

$$\sin(x) = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots+(-1)^n\frac{1}{n!}x^{2n+1}+\cdots.$$

Thus,

$$\begin{aligned} \frac{\sin(x)-x}{x^3} &= \frac{1}{x^3}\left(-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots+(-1)^n\frac{1}{n!}x^{2n+1}+\cdots\right) \\ &= -\frac{1}{6}+\frac{1}{5!}x^2-\frac{1}{7!}x^4+\cdots+(-1)^n\frac{1}{n!}x^{2n-2}+\cdots \end{aligned}$$