Last Name:
First Name:
Instructor:

Math 151
Group Final (Spring 2007)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- **1** /5
- **2** /5
- **3** /5
- **4** /5
- **5** /5
- **6** /10
- **7** /10
- **8** /10
- **9** /10
- **10** /5
- **11** /10
- **12** /10
- **13** /5
- **14** /5

Total: **/100**

1 (5 pts.) Determine

$$\int x \cosh(5x)\, dx$$

2 (5 pts.) Determine

$$\int \frac{x+26}{x^2-2x-8}dx$$

3 (5 pts.) Determine

$$\int \cos^2\left(\frac{3x}{4}\right) dx.$$

4 (5 pts.) Determine whether the improper integral

$$\int_0^\infty \frac{1}{16+4x^2}dx$$

converge or diverges, and its value in case it converges.

5 (5 pts.) Use a comparison test to determine whether the improper integral

$$\int_1^\infty \frac{e^{-x^2}}{x^4} dx$$

converges or diverges. You need not determine the value of the improper integral in case of convergence.

6 (10 pts.) Determine the area of the surface that is obtained by revolving the graph of

$$f(x) = x^3$$

and the interval $[0, 2]$ about the x-axis.

7 (10 pts.) Determine the solution of the initial value problem

$$\frac{dy}{dt} = -\frac{1}{4}y\left(t\right) + 3t, \ \ y\left(4\right) = 2.$$

8 (10 pts.) Determine the solution of the initial value problem

$$\frac{dy}{dt} = \frac{ty^2}{\sqrt{1+t^2}},\ y(0) = 3$$

9. Let

$$r = f(\theta) = \cos(2\theta).$$

a) (5 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a local maximum or minimum value.

b) (5 pts.) Sketch the graph of $r = f(\theta)$, where $0 \leq \theta \leq 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

10 (5 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n}{n!}$$

converges absolutely, converges conditionally or diverges.

11 (10 pts.) Determine whether the infinite series

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$$

converges absolutely, converges conditionally or diverges.

12 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2^n} (x+6)^n .$$

(You need not investigate the series at the endpoints of the interval.)

13 (5 pts.) Let $f(x) = \sin(x)$. Determine the part of the Taylor series for f based at $\pi/6$ up to the term that has $(x - \pi/6)^3$.

14 (5 pts.) Given that

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$$

Determine the Maclaurin series for $\arcsin(x)$ up to the term that has x^7.

Math 151
Group Final (Spring 2007)
Solutions

1. We set $u = x$ and $dv = \cosh(5x)\,dx$ so that

$$du = dx \text{ and } v = \int \cosh(5x)\,dx = \frac{1}{5}\sinh(5x).$$

Thus,

$$\begin{aligned}\int x\cosh(5x)\,dx &= \int u dv \\ &= uv - \int v du \\ &= x\left(\frac{1}{5}\sinh(5x)\right) - \frac{1}{5}\int \cosh(5x)\,dx \\ &= \frac{1}{5}x\sinh(5x) - \frac{1}{25}\cosh(5x) + C\end{aligned}$$

2.

$$\frac{x+26}{x^2-2x-8} = \frac{x+26}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$\Leftrightarrow$

$$x + 26 = A(x+2) + B(x-4).$$

We set $x = 4$:

$$30 = 6A \Rightarrow A = 5.$$

We set $x = -2$:

$$24 = -6B \Rightarrow B = -4.$$

Thus.

$$\frac{x+26}{x^2-2x-8} = \frac{5}{x-4} - \frac{4}{x+2}$$

Therefore.

$$\int \frac{x+26}{x^2-2x-8}dx = -4\ln(|x+2|) + 5\ln(|x-4|) + C$$

3. We have

$$\cos^2\left(\frac{3x}{4}\right) = \frac{1+\cos\left(\frac{3x}{2}\right)}{2}.$$

Therefore,

$$\begin{aligned}\int \cos^2\left(\frac{3x}{4}\right)dx &= \int \frac{1+\cos\left(\frac{3x}{2}\right)}{2}dx \\ &= \frac{1}{2} + \frac{1}{3}\sin\left(\frac{3x}{2}\right) + C \quad \left(= \frac{1}{2}x + \frac{2}{3}\cos\left(\frac{3}{4}x\right)\sin\left(\frac{3}{4}x\right) + C\right)\end{aligned}$$

4.

$$\int \frac{1}{16+4x^2}dx = \int \frac{1}{16\left(1+\left(\frac{x}{2}\right)^2\right)}dx = \frac{1}{8}\arctan\left(\frac{1}{2}x\right)$$

Therefore,

$$\int_0^b \frac{1}{16+4x^2}dx = \frac{1}{8}\arctan\left(\frac{1}{2}b\right)$$

Thus,

$$\int_0^\infty \frac{1}{16+4x^2}dx = \lim_{b\to\infty}\left(\frac{1}{8}\arctan\left(\frac{1}{2}b\right)\right) = \frac{1}{16}\pi$$

5. If $x > 0$

$$0 < \frac{e^{-x^2}}{x^4} < \frac{1}{x^4}$$

and

$$\int_1^\infty \frac{1}{x^4}dx$$

converges. Therefore the given integral converges.

6. We set $u = 1+9x^4$ so that $du = 36x^3dx$. Thus,

$$\begin{aligned}\int 2\pi x^3\sqrt{1+9x^4}dx &= \pi\int \frac{1}{18}u^{1/2}du \\ &= \frac{\pi}{18}\left(\frac{u}{3/2}^{3/2}\right) = \frac{1}{27}\pi\left(1+9x^4\right)^{\frac{3}{2}}\end{aligned}$$

Therefore, the area of the surface is

$$\begin{aligned}\int_0^2 2\pi x^3\sqrt{1+9x^4}dx &= \frac{1}{27}\pi\left(1+9x^4\right)^{\frac{3}{2}}\Big|_0^2 \\ &= \frac{145}{27}\sqrt{145}\pi - \frac{1}{27}\pi\end{aligned}$$

7. The integrating factor is $e^{t/4}$:

$$\begin{aligned}\frac{dy}{dt}+\frac{1}{4}y(t) &= 3t; \\ \frac{d}{dt}\left(e^{t/4}y(t)\right) &= 3te^{t/4}; \\ e^{t/4}y(t) &= \int 3te^{t/4}dt = 12e^{\frac{1}{4}t}(t-4)+C; \\ y(t) &= 12t-48+Ce^{-t/4}; \\ y(4)=2 &\Leftrightarrow 2 = 48-48+Ce^{-1} \\ &\Leftrightarrow C = 2e;\end{aligned}$$

$$y(t) = 12t-48+2ee^{-t/4}$$

8.

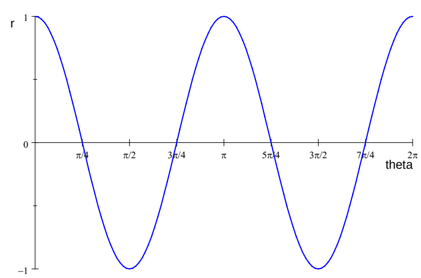
$$\begin{aligned}
\int \frac{1}{y^2} dy &= \int \frac{t}{\sqrt{1+t^2}} dt; \\
-\frac{1}{y} &= \sqrt{t^2+1} + C; \\
y(t) &= -\frac{1}{\sqrt{1+t^2}+C}; \\
y(0) &= 3 \Leftrightarrow 3 = -\frac{1}{1+C} \Leftrightarrow C+1 = -\frac{1}{3} \Leftrightarrow C = -\frac{4}{3};
\end{aligned}$$

$$y(t) = \frac{1}{-\sqrt{1+t^2}+\frac{4}{3}}$$

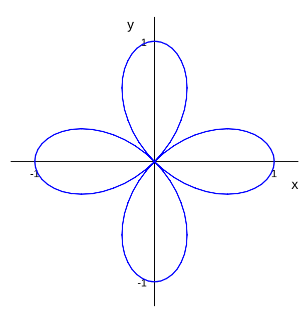
9.

a)

$$\begin{aligned}
\cos(2\theta) &= 0 \text{ if } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \\
\cos(2\theta) &= 1 \text{ if } \theta = 0, \pi, 2\pi, \\
\cos(2\theta) &= -1 \text{ if } \theta = \frac{\pi}{2}, \frac{3\pi}{2},
\end{aligned}$$



b)



13. We have

$$f(x) = \sin(x), \ f'(x) = \cos(x), \ f''(x) = -\sin(x), \ f^{(3)}(x) = -\cos(x).$$

Therefore,

$$\begin{aligned} f\left(\frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \\ f'\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \\ f''\left(\frac{\pi}{6}\right) &= -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}, \\ f^{(3)}\left(\frac{\pi}{6}\right) &= -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} \sin(x) &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{1}{6}\pi\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(x - \frac{1}{6}\pi\right)^2 + \frac{1}{3!}\left(-\frac{\sqrt{3}}{2}\right)\left(x - \frac{1}{6}\pi\right)^3 + \cdots \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{1}{6}\pi\right) - \frac{1}{4}\left(x - \frac{1}{6}\pi\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{1}{6}\pi\right)^3 + \cdots \end{aligned}$$

14. By the Fundamental Theorem of Calculus

$$\begin{aligned} \arcsin(x) = \arcsin(x) - \arcsin(0) &= \int_0^x \frac{d}{dt}\arcsin(t)\,dt \\ &= \int_0^x \frac{1}{\sqrt{1-t^2}}dt \\ &= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \frac{5}{16}t^6 + \cdots\right)dt \\ &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots \end{aligned}$$