Study Guide and Sample Problems for the Fall 2005 Math 150 Group Final

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Questions on the group final will be similar to the sample problems provided in this document. There will be approximately 20 questions. No books, notes, calculators, cell phones or personal stereos are allowed.

Questions will be selected from the following list of topics:

Basic Calculus Skills:

- **Finite or infinite limits at a point or at infinity, L'Hospital's rule**
- **Differentiation of combinations of powers of x, trigonometric, exponential and logarithmic functions**
- **Indefinite and definite integrals, the substitution rule**

Other Topics

- **Differentiation directly from the definition**
- **Local linear approximations and the differential**
- **Related rates**
- **Implicit differentiation**
- **Monotonicity, local and absolute extrema and applications**
- **Concavity**
- **Curve sketching**
- **The Fundamental Theorem of Calculus**
- **One-dimensional motion**
- **Area between two curves**

In preparation for the group final, students should memorize the following **differentiation formulas**.

1. $\frac{d}{dx}x^r = rx^{r-1}$
2. $\frac{d}{dx}\sin(x) = \cos(x)$
3. $\frac{d}{dx}\cos(x) = -\sin(x)$
4. $\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$
5. $\frac{d}{dx}a^x = \ln(a)\,a^x$
6. $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$
7. $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
8. $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$
9. $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

Students should memorize the following **antidifferentiation formulas**.

1. $\int x^r dx = \frac{1}{r+1}x^{r+1}\ \ (r \neq -1)$
2. $\int \frac{1}{x}dx = \ln(|x|)$
3. $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$
4. $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$
5. $\int e^{ax}dx = \frac{1}{a}e^{ax}$
6. $\int a^x dx = \frac{1}{\ln(a)}a^x$
7. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$
8. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right)$

Remark: Students should be able to compute the exact **values of the trigonometric functions at special angles** such as $\pi/3$, $\pi/6$, $\pi/4$, $2\pi/3$, $-3\pi/4$ and exact **values of the inverse trigonometric functions** such as $\arctan\left(\sqrt{3}\right)$, $\arcsin(-1/2)$, $\arccos(0)$.

SAMPLE PROBLEMS for the GROUP FINAL

1. Determine the finite or infinite limit or state that the limit does not exist in either the finite or infinite sense:

a)
$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$$

b)
$$\lim_{x\to 0} \frac{\sin(2x)}{x}$$

c)
$$\lim_{x\to 0} \frac{\cos(x)-1}{x}$$

d)
$$\lim_{x\to +\infty} \frac{x^2+3x-1}{4x^2-7}$$

e)
$$\lim_{x\to 2-} \frac{1}{x-2}$$

f)
$$\lim_{x\to -3} \frac{2}{x+3}$$

g)
$$\lim_{x\to 0+} \sqrt{x}\ln(x)$$
(using L'Hospital's rule)

h)
$$\lim_{x\to +\infty} x^2 e^{-x}$$
(using L'Hospital's rule)

i)
$$\lim_{x\to +\infty} \left(1-\frac{3}{x}\right)^x$$
(using L'Hospital's rule)

2. Determine the derivative:

a)
$$\frac{d}{dx}\left(x^2\cos(x)\right)$$

b)
$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

c)
$$\frac{d}{dx}\sin\left(x^2\right)$$

d)
$$\frac{d}{dx}\sqrt{4+x^2}$$

e)
$$\frac{d}{dx}\left(x^2 e^{-x/10}\right)$$

f)
$$\frac{d}{dx}\left(\frac{1}{2}\sin(2x)+\frac{1}{3}\cos(3x)\right)$$

g)
$$\frac{d}{dx}\arcsin(x/4)$$

h)
$$\frac{d}{dx}\ln\left(\frac{x}{x^3+1}\right)$$

i)
$$\frac{d}{dx}\int_0^x e^{(t^2)}dt$$

j)
$$\frac{d}{dx}\sin\left(\frac{x}{2+x}\right)\Bigg|_{x=0}$$

k)
$$\frac{d}{dx}\ln\left(\sqrt{x}\right)\Big|_{x=1}$$

l)
$$\frac{d}{dx}\arccos(2x)\Big|_{x=0}$$

3. Determine the indefinite or definite integral:

a)
$$\int x \sin\left(3x^2\right) dx$$

b)
$$\int x\sqrt{x^2+4}dx$$

c)
$$\int 10^{-4x}dx$$

d)
$$\int \frac{2}{3-4x}dx$$

e)
$$\int \frac{1}{(2-3x)^2}dx$$

f)
$$\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}dx$$

g)
$$\int \frac{x}{x^2+9}dx$$

h)
$$\int \frac{1}{4x^2+9}dx$$

i)
$$\int e^{-x/2}dx$$

j)
$$\int \frac{1}{\sqrt{9-4x^2}}dx$$

k)
$$\int_{\pi/3}^{\pi/4} \sin(x)\, dx$$

l)
$$\int_{-3}^{3} \frac{1}{x^2+9}dx$$

m)
$$\int_{1}^{e} \frac{\ln(x)}{x}dx$$

n)
$$\int_{0}^{1} \left(\frac{d}{dx}e^{-x^2}\right) dx$$

4. Let

$$f(x) = \frac{1}{x^2 + x}.$$

Determine $f'(x)$ **directly from the definition** of the derivative of f at x.

5, Let

$$f(x) = \frac{1}{\sqrt{x}}.$$

a) Determine L, the linear approximation to f based at 4.
b) Make use of L to approximate

$$\frac{1}{\sqrt{3.9}}$$

(you can leave your answer in terms of fractions

6. Let $f(x) = \tan(x)$.
a) Determine the differential df.
b) Make use of df to approximate

$$\tan\left(\frac{\pi}{4} - 0.1\right).$$

7. Assume that helium is being pumped into a spherical balloon at a constant rate of 100 cubic centimeters per second. Also assume that the shape of the balloon is a perfect sphere as it is being inflated. Determine the rate at which the radius of the balloon is increasing at the instant its radius is 10 centimeters.

8. An airplane is flying at an altitude of 2 miles with a speed of 200 miles/hour. It is being tracked by an observer on the ground with a searchlight. Find the rate at which the angle θ between the searchlight and the vertical direction changes at the instant the horizontal distance of the plane from the observer is 10 miles.

9. Assume that a ladder which is 10 feet long is leaning against a wall and its base is sliding away from the wall at the rate of 2 ft/sec. Determine the rate at which the top of the ladder is sliding down the wall at the instant the base of the ladder is 4 feet from the wall.

10. Consider the equation

$$y^5 + y^2 - y - x^2 + 1 = 0.$$

a) Assume that $y(x)$ represents a function that is defined implicitly by the given equation. Determine $y'(x)$.
b) Evaluate $y'(1)$ if $y(1) = 0$. Determine the tangent line to the graph of the equation at $(1, 0)$.

11. Let

$$f(x) = x^2 + \frac{1}{x}$$

a) Determine the intervals on which f is increasing/decreasing and the points at which f has a local maximum or minimum.
b) Determine the absolute maximum and the absolute minimum of f on $(0, +\infty)$, provided that such values exist. You have to justify a claim of nonexistence.

12. Let $f(x) = x(x-3)^{2/3}$. Determine the absolute maximum and minimum of f on the interval $[0, 3]$.

13. Let

$$f(x) = \frac{e^{x/2}}{x^2}$$

Determine the subintervals of $(0, +\infty)$ on which f is increasing/decreasing and the absolute maximum and the absolute minimum of f on $(0, +\infty)$, provided that such values exist. You have to justify a claim of nonexistence.

14. Let

$$f(x) = x + \frac{1}{x - 2}$$

a) Determine the vertical asymptotes for the graph of f and the relevant infinite limits.
b) Show that the line $y = x$ is an oblique asymptote for the graph of f at $\pm\infty$.
c) Determine the intervals on which f is increasing/decreasing and the points at which f has a local maximum or local minimum.
d) Sketch the graph of f. Indicate the asymptotes clearly.

15. Let

$$f(x) = -2x - \frac{1}{2}x^2 + \frac{1}{3}x^3.$$

Determine the intervals on which the graph of f is concave up/concave down and the points of inflection on the graph of f.

16. Let

$$f(x) = -8x - x^2 + \frac{1}{3}x^3.$$

Use the second derivative test to determine the points at which f has a local maximum or a local minimum.

17. Let

$$f(x) = x^2 e^{-x}$$

a) Determine $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to 0+} f(x)$.
b) Determine the intervals on which f is increasing/decreasing
c) Determine the x-coordinates of the points of inflection on the graph of f and the intervals on which the graph of f is concave up/concave down.
d) Sketch the graph of f.

18 Determine the dimensions of the rectangle that has the greatest area among all rectangles which are inscribed in a circle of radius 1.

19. Determine $y(x)$ if

$$\frac{dy}{dx} = \frac{1}{x^2+1} \text{ and } y(1) = \pi$$

20. Assume that $v(t) = \cos(4t)$ is the velocity at the instant t of an object in one-dimensional motion and let $f(t)$ be its position at time t. Assume that $f(\pi/8) = -2$. Determine $f(t)$.

21 Let $f(x) = x^2 - 2x - 1$ and $g(x) = -x^2 + 2x + 5$.
a) Sketch the region G between the graph of f, the graph of g, the line $x = 1$ and the line $x = 5$.
b) Calculate the area of G.