Last Name:
First Name:
Instructor:

Math 151
Group Final (Fall 2005)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- **1** /5
- **2.** /10
- **3** /5
- **4.** /5
- **5** /5
- **6.** /5
- **7.** /10
- **8.** /10
- **9** /10
- **10** /5
- **11** /10
- **12** /10
- **13** /5
- **14** /5

Total: /100

1.(5 pts.) Determine

$$\int \frac{x-11}{x^2+3x-4}dx$$

2 (10 pts.) Determine

$$\int \frac{4x^2 + 2x + 19}{(x-2)\left(x^2+9\right)} dx$$

3 (5 pts.) Determine

$$\int xe^{-x/2}dx$$

4 (5 pts.) Does the improper integral

$$\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$$

converge or diverge? Justify your response.
Hint: Make use of a comparison theorem.

5. (5 pts.) Determine whether the improper integral

$$\int_2^4 \frac{1}{x-2}\,dx$$

converges or diverges, and the value of the improper integral in case of convergence.

6. (5 pts.) Determine the volume of the solid that is obtained by revolving the graph of

$$f(x) = \cos\left(\frac{x}{2}\right)$$

on the interval $[-\pi, \pi]$ about the x-axis.

7. (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem

$$\frac{dy}{dt} + 2ty\left(t\right) = 2te^{-t^2}, \;\; y\left(2\right) = 1.$$

8. (10 pts.) Determine the solution of the initial value problem

$$\frac{dy}{dx} = y^2 e^{-x}, \ y(0) = 2$$

9. Let

$$r = f(\theta) = 1 - 2\cos(\theta).$$

a) (5 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a maximum or minimum value.

b) (5 pts.) Sketch the graph of $r = f(\theta)$, where $0 \leq \theta \leq 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

10. (5 pts.) Determine whether the infinite series

$$\sum_{n=0}^{\infty} (-1)^n \, \frac{3^{2n+1}}{(2n+1)!}$$

converges absolutely, converges conditionally or diverges.

11.(10 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$$

converges absolutely, converges conditionally or diverges.

12. (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{4^n n^{1/3}}.$$

(You need not investigate the series at the endpoints of the interval.)

13. (5 pts.) Determine the Maclaurin series of

$$f(x) = \frac{1}{1+x^2}$$

and the open interval of convergence of the series (Display the first 4 terms and the term that involves x^{2n} for an arbitrary positive integer n).
Hint: Think of the geometric series.

14. (5 pts.) Let $f(x) = \sin(x)$. Determine the first 4 terms of the Taylor series of f in powers of $(x - \pi/6)$.

Solutions

1.

$$\frac{x-11}{x^2+3x-4}=\frac{x-11}{(x+4)(x-1)}=\frac{A}{x+4}+\frac{B}{x-1}$$

$\Leftrightarrow$

$$x-11=A(x-1)+B(x+4)$$

Set $x=-4:$

$$-15=-5A\Rightarrow A=3.$$

Set $x=1:$

$$-10=5B\Rightarrow B=-2.$$

Therefore,

$$\int\frac{x-11}{x^2+3x-4}dx=\int\frac{3}{x+4}dx-\int\frac{2}{x-1}dx=3\ln(|x+4|)-2\ln(|x-1|).$$

2. We set

$$\frac{4x^2+2x+19}{(x-2)(x^2+9)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+9}$$

Thus,

$$\begin{aligned}4x^2+2x+19 &= A\left(x^2+9\right)+(Bx+C)(x-2)\\ &= (A+B)x^2+(-2B+C)x+(9A-2C)\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}A+B &= 4\\ -2B+C &= 2\\ 9A-2C &= 19\end{aligned}$$

$\Leftrightarrow$

$$A=3,\ B=1,\ C=4.$$

Therefore,

$$\frac{4x^2+2x+19}{(x-2)(x^2+9)}=\frac{3}{x-2}+\frac{x+4}{x^2+9}$$

Thus,

$$\int\frac{4x^2+2x+19}{(x-2)(x^2+9)}dx=3\ln(|x-2|)+\frac{1}{2}\ln\left(x^2+9\right)+\frac{4}{3}\arctan\left(\frac{x}{3}\right).$$

3. We set $u = x$ and $dv = e^{-x/2}$ so that

$$du = dx \text{ and } v = \int e^{-x/2}dx = -2e^{-x/2}.$$

Therefore,

$$\begin{aligned}\int xe^{-x/2}dx = \int udv &= uv - \int vdu \\ &= x\left(-2e^{-x/2}\right) + 2\int e^{-x/2}dx \\ &= -2xe^{-x/2} - 4e^{-x/2}\end{aligned}$$

4. We have

$$0 \le \frac{\sin^2(x)}{x^2} \le \frac{1}{x^2}$$

and

$$\int_1^\infty \frac{1}{x^2}$$

converges. Therefore, the given integral converges as well.

5. The integral is improper since

$$\lim_{x\to 2+} \frac{1}{x-2} = +\infty.$$

If $0 < \varepsilon < 2$,

$$\int_{2+\varepsilon}^4 \frac{1}{x-2}dx = \ln\left(|x-2|\right)\Big|_{2+\varepsilon}^4 = \ln(2) - \ln(\varepsilon).$$

Therefore,

$$\lim_{\varepsilon\to 0+} \int_{2+\varepsilon}^4 \frac{1}{x-2}dx = \lim_{\varepsilon\to 0+} \left(\ln(2) - \ln(\varepsilon)\right) = +\infty.$$

Thus, the given improper integral diverges.

6. The volume is

$$\int_{-\pi}^{\pi} \pi\cos^2\left(\frac{x}{2}\right)dx = \pi\int_{-\pi}^{\pi}\cos^2\left(\frac{x}{2}\right)dx$$

We have

$$\int \cos^2\left(\frac{x}{2}\right)dx = \int \frac{1+\cos(x)}{2}dx = \frac{x}{2} + \frac{1}{2}\sin(x).$$

Therefore,

$$\int_{-\pi}^{\pi} \pi\cos^2\left(\frac{x}{2}\right)dx = \pi\left(\frac{x}{2} + \frac{1}{2}\sin(x)\Big|_{x=-\pi}^{x=\pi}\right) = \pi\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \pi^2.$$

7. The integrating factor is

$$e^{\int 2tdt} = e^{t^2}.$$

Thus,

$$e^{t^2}\left(\frac{dy}{dt} + 2ty(t)\right) = e^{t^2}\left(2te^{-t^2}\right)$$

$\Rightarrow$

$$\frac{d}{dt}\left(e^{t^2}y(t)\right) = 2t$$

$\Rightarrow$

$$e^{t^2}y(t) = t^2 + C \Rightarrow y(t) = t^2e^{-t^2} + Ce^{-t^2}.$$

We have

$$y(2) = 1 \Leftrightarrow 4e^{-4} + Ce^{-4} = 1 \Leftrightarrow C = e^4 - 4.$$

Therefore, the solution of the given initial value problem is

$$y(t) = t^2e^{-t^2} + \left(e^4 - 4\right)e^{-t^2}.$$

8.

$$\begin{aligned}\frac{dy}{dx} = y^2e^{-x} \Rightarrow \frac{1}{y^2}dy = e^{-x}dx \quad &\Rightarrow \quad \int y^{-2}dy = \int e^{-x}dx \\ &\Rightarrow \quad -\frac{1}{y} = -e^{-x} + C \\ &\Rightarrow \quad y(x) = \frac{1}{e^{-x} - C}\end{aligned}$$

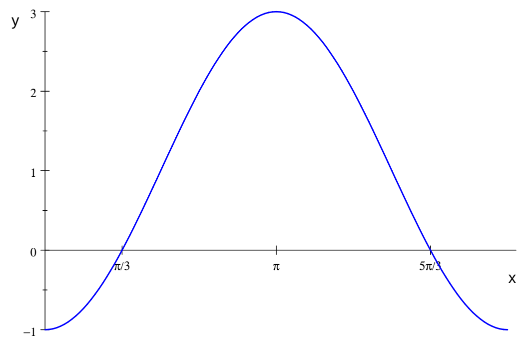
We have

$$y(0) = 2 \Leftrightarrow \frac{1}{1 - C} = 2 \Leftrightarrow 1 - C = \frac{1}{2} \Longleftrightarrow C = \frac{1}{2}$$

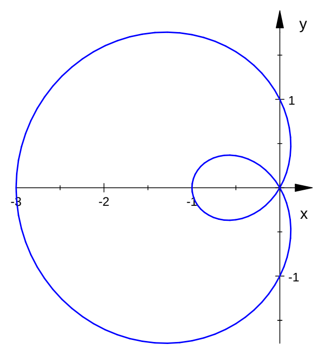
Therefore,

$$y(x) = \frac{1}{e^{-x} - \frac{1}{2}}$$

9.
a)



b)



10. We have

$$\lim_{n\to\infty} \frac{\dfrac{3^{2n+3}}{(2n+3)!}}{\dfrac{3^{2n+1}}{(2n+1)!}} = \lim_{n\to\infty} \left(\frac{3^2}{(2n+2)\,(2n+3)} \right) = 0 < 1.$$

By the ratio test, the series converges absolutely.

11. The series

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

diverges. Indeed, we can use the divergent series $\sum 1/n$ as a "comparison series". We have

$$\lim_{n\to\infty} \frac{\dfrac{n}{n^2+1}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0,$$

so that $\sum n/\left(n^2+1\right)$ diverges, by the limit-comparison test (alternatively, one can apply the integral test.

On the other hand, the series is an alternating series and

$$\lim_{n\to\infty} \frac{n}{n^2+1} = 0.$$

The sequence

$$\frac{n}{n^2+1},\ n = 1, 2, 3, \ldots$$

is monotone decreasing. Indeed, if we set

$$f(x) = \frac{x}{x^2+1},$$

then

$$f'(x) = \frac{1-x^2}{\left(x^2+1\right)^2} < 0$$

if $x > 1$. Thus, the theorem on alternating series is applicable, so that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$$

converges. Since the series does not converge absolutely, it converges conditionally.

12. We have

$$\lim_{n\to\infty} \left| \frac{(x-3)^n}{4^n n^{1/3}} \right|^{1/n} = \frac{|x-3|}{4} \left(\frac{1}{\lim_{n\to\infty} \left(n^{1/n}\right)^{1/3}} \right) = \frac{|x-3|}{4} < 1$$

iff $|x-3| < 4$. Therefore, the radius of convergence is 4 and the open interval of convergence is $(-1, 7)$.

13.

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)}\\ &= 1+\left(-x^2\right)+\left(-x^2\right)^2+\left(-x^2\right)^3++\cdots+\left(-x^2\right)^n+\cdots\\ &= 1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots\end{aligned}$$

The series converges iff

$$\left|-x^2\right|<1 \Leftrightarrow |x|^2<1 \Leftrightarrow |x|<1.$$

Thus, the open interval of the series is $(-1,1)$.

14. We have

$$f(x)=\sin(x),\ f'(x)=\cos(x),\ f''(x)=-\sin(x), f^{(3)}(x)=-\cos(x).$$

Therefore,

$$f\left(\frac{\pi}{6}\right)=\frac{1}{2},\ f'\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2},\ f''\left(\frac{\pi}{6}\right)=-\frac{1}{2},\ f^{(3)}\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}.$$

Thus,

$$\sin(x)=f(x)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3+\cdots$$